Stellar Relaxation Time

[Chandrasekhar 1960, *Principles of Stellar Dynamics,* Chap II]
[Ostriker & Davidson 1968, *Ap.J.,* **151,** 679]

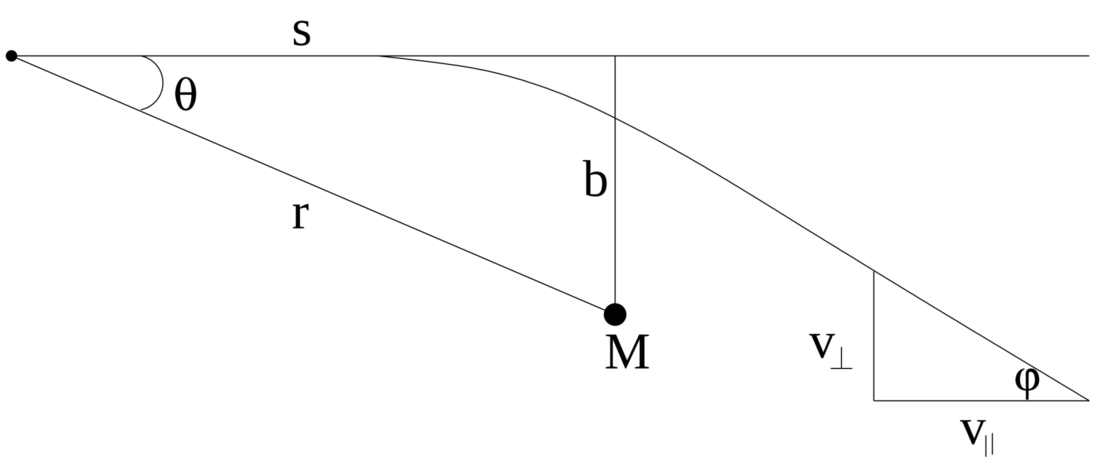
Do stars ever collide? Are interactions between stars (as opposed to the general system potential) important? We can answer this question by calculating the time it takes for a star's orbit to be "significantly" perturbed by individual encounters with other stars. To calculate this *relaxation time*, let's first define the word "significant". One way of doing this is through total energy: when does the kinetic energy exchanged during stellar encounters equal the star's original kinetic energy, *i.e.,*

$$T_E \Longrightarrow \sum (\Delta E)^2 = E \tag{30.01}$$

But for simplicity, we'll define "significant" as the time it takes a star to lose all memory of its original trajectory, *i.e.,*

$$T_D \Longrightarrow \sum \sin^2 \varphi = 1 \tag{30.02}$$

We then assume that a) all deflections are two-body encounters, b) each encounter is statistically independent, and c) close encounters are insignificant compared to long-range encounters, so that during each encounter, $|\Delta E| \ll E$. Under these assumptions, all the deflections are small ($\sin \varphi \ll 1$), and we can use the Born approximation, where ($v_{\rm init} \approx v_{\rm final} \approx v$).



For a single encounter, the deflection angle, φ is related to the initial impact parameter, b, by

$$F_\perp = m\frac{dv_\perp}{dt} \implies v_\perp = \int_{-\infty}^{\infty} dv_\perp = \frac{1}{m}\int_{-\infty}^{\infty} F_\perp dt \qquad (30.03)$$

From the geometry of the encounter

$$F_\perp = F\sin\theta = F\left(\frac{b}{r}\right) = \left(\frac{G\mathcal{M}m}{r^2}\right)\left(\frac{b}{r}\right) \qquad (30.04)$$

Also, from the Born approximation

$$v_\parallel dt = vdt = ds \implies dt = \frac{ds}{v} \qquad (30.05)$$

So

$$v_\perp = \frac{1}{m}\int_{-\infty}^{\infty} F_\perp dt = \frac{2}{m}\int_0^{\infty}\left(\frac{G\mathcal{M}m}{r^2}\right)\left(\frac{b}{r}\right)\left(\frac{1}{v}\right) ds \qquad (30.06)$$

or, since $r = (s^2 + b^2)^{1/2}$

$$v_\perp = \frac{2G\mathcal{M}}{v}\int_0^{\infty}\frac{b}{(s^2+b^2)^{3/2}}\,ds = \frac{2G\mathcal{M}}{v}\int_0^{\infty}\frac{ds/b}{(1+(s/b)^2)^{3/2}} \qquad (30.07)$$

Letting $x = s/b$

$$v_\perp = \frac{2G\mathcal{M}}{vb}\int_0^{\infty}\frac{dx}{(1+x^2)^{3/2}} = \frac{2G\mathcal{M}}{vb}\cdot\frac{x}{(1+x^2)^{1/2}}\Bigg|_0^{\infty}$$

$$= \frac{2G\mathcal{M}}{vb} \qquad (30.08)$$

Since for small deflections, $\tan\varphi \approx \varphi = v_\perp / v$

$$\varphi = \frac{2G\mathcal{M}}{v^2 b} \tag{30.09}$$

Now, let's sum this over all possible collisions. The number of collisions that take place in time dt depends on the impact parameter, the distance a star travels in dt, and the density of stars in the stellar system, N, *i.e.*,

$$N_{\text{coll}} = (2\pi b\, db) \cdot (vdt) \cdot N \tag{30.10}$$

So, to deflect the star by 90°,

$$\begin{aligned}
\sum \sin^2\varphi \approx \sum \varphi^2 = 1 = &\int_0^{T_D}\int_{b_{\min}}^{b_{\max}} (2\pi b\, db)(vdt)\, N \cdot \varphi^2 \\
&= \int_0^{T_D}\int_{b_{\min}}^{b_{\max}} (2\pi b\, db)(vdt)N \cdot \left(\frac{2G\mathcal{M}}{v^2 b}\right)^2 \\
&= \frac{8\pi G^2 \mathcal{M}^2 N}{v^3} T_D \int_{b_{\min}}^{b_{\max}} \frac{db}{b}
\end{aligned} \tag{30.11}$$

As for the limits on the log quantity, we can use the obvious fact that no deflection angle can be greater than π. Thus

$$\varphi = \frac{2G\mathcal{M}}{v^2 b_{\min}} = \pi \Longrightarrow\ b_{\min} = \frac{2G\mathcal{M}}{\pi v^2} \tag{30.12}$$

Similarly, it is clear that the maximum impact angle must be less than the mean distance between stars, so

$$N = \frac{1}{(4/3)\pi\, b_{max}^3} \Longrightarrow\ b_{\max} = \left(\frac{3}{4\pi N}\right)^{1/3} \tag{30.13}$$

So

$$\left(\frac{8\pi G^2\mathcal{M}^2 N}{v^3}\right) T_D \ln\left\{\frac{b_{\rm max}\pi v^2}{2G\mathcal{M}}\right\} = 1 \tag{30.14}$$

giving

$$T_D = \frac{v^3}{8\pi G^2\mathcal{M}^2 N} \ln\left\{\frac{b_{\rm max} v^2 \pi}{2G\mathcal{M}}\right\} \tag{30.15}$$

Obviously, the above derivation involves a number of approximations. A more rigorous derivation by Chandrasekhar gives

$$T_D = \frac{v^3}{8\pi G^2\mathcal{M}^2 N H(\chi)} \ln\left\{\frac{b_{\rm max} v^2}{2G\mathcal{M}}\right\} \tag{30.16}$$

and

$$T_E = \frac{v^3}{32\pi G^2\mathcal{M}^2 N G(\chi)} \ln\left\{\frac{b_{\rm max} v^2}{2G\mathcal{M}}\right\} \tag{30.17}$$

where $H(\chi)$ and $G(\chi)$ are factors of the order unity that depend on the stellar distribution function. Finally, Ostriker & Davidson (1968) give an improved, recursive expression for the relaxation time

$$T_P = \frac{v^3}{8\pi G^2\mathcal{M}^2 N} \ln\left\{\frac{v^3 T_P}{2G\mathcal{M}}\right\} \tag{30.17}$$

In the solar neighborhood, this timescale is much larger than a Hubble time. Thus, the motions of stars are "collisionless", and controlled only by the overall Galactic potential.

The Collisionless Boltzmann Equation

[Binney & Tremaine 1987, *Galactic Dynamics*, 1987]

The basis of understanding galactic dynamics lies with the collisionless Boltzmann equation. Imagine a closed volume, V, bounded by a surface S, and containing a mass, $\mathcal{M}(t)$. The net amount of mass flowing through a surface is equal to the change of mass in the volume, *i.e.*,

$$\int_S \rho\, v\, dS = -\frac{d\mathcal{M}}{dt} = \int_V \frac{\partial \rho}{\partial t}\, dV \tag{30.18}$$

But the divergence theorem in mathematics says

$$\int_S Q\, dS = \int_V \nabla \cdot Q\, dV \tag{30.19}$$

so

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla \cdot (\rho v)\, dV = 0 \tag{30.20}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \tag{30.21}$$

Now expand this concept to stars flowing in a 6-dimensional phase space (x, y, z, v_x, v_y, v_z). Assume that the flow of stars is smooth, and is controlled by a potential per unit mass Φ. Now let

$$
\begin{aligned}
f &= \text{ the stellar density in the 6-D space} \\
\vec{\omega} &= \text{ the 6-D position coordinate} \\
\vec{x} &= \text{ the 3-D space coordinate} \\
\vec{v} &= \text{ the 3-D velocity coordinate}
\end{aligned}
$$

With these definitions, the continuity equation for a fluid of stars is

$$\frac{\partial f}{\partial t} + \nabla \cdot (f \dot{\omega}) = 0 \tag{30.22}$$

Now let's expand this out to

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \left(\dot{\omega}_\alpha \frac{\partial f}{\partial \omega_\alpha} + f \frac{\partial \dot{\omega}_\alpha}{\partial \omega_\alpha} \right) = 0 \tag{30.23}$$

and explicitly write out the space and velocity parts of the second part of the equation.

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{\omega}_\alpha \frac{\partial f}{\partial \omega_\alpha} + \sum_{i=1}^{3} f \frac{\partial \dot{x}_i}{\partial x_i} + \sum_{i=1}^{3} f \frac{\partial \dot{v}_i}{\partial v_i} = 0 \tag{30.24}$$

Since $\vec{x}$ and $\vec{v}$ are independent dimensions,

$$\frac{\partial \dot{x}_i}{\partial x_i} = \frac{\partial v_i}{\partial x_i} = 0 \tag{30.25}$$

Also, for conservative forces with no collisions

$$\dot{v} = -\frac{\partial \Phi}{\partial x} \tag{30.26}$$

so

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{\omega}_\alpha \frac{\partial f}{\partial \omega_\alpha} + \sum_{i=1}^{3} f \frac{\partial}{\partial v_i} \left\{ -\frac{\partial \Phi}{\partial x_i} \right\} = 0 \tag{30.27}$$

Also, since the potential a particle feels depends only on its location, and not on its velocity

$$\frac{\partial}{\partial v_i} \left(\frac{\partial \Phi}{\partial x_i} \right) = 0 \tag{30.28}$$

which leaves only

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{\omega}_\alpha \frac{\partial f}{\partial \omega_\alpha} = 0 \tag{30.29}$$

Finally, to make the equation more transparent, we can again explicitly write out the position and velocity parts of the sum

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \dot{x}_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{3} \dot{v}_i \frac{\partial f}{\partial v_i} = 0 \tag{30.30}$$

or, if we substitute $\dot{x} = v$ and $\dot{v} = -\frac{\partial \Phi}{\partial x}$,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left\{ v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right\} = 0 \tag{30.31}$$

which, in vector notation is

$$\frac{\partial f}{\partial t} + (\vec{v} \cdot \nabla f) - \left(\nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} \right) = 0 \tag{30.32}$$

This is sometimes called the collisionless Boltzmann equation, the Vlasov equation, or the equation of continuity. Note that this equation, when expressed in terms of a Lagrangian derivative (*i.e.,* from the point of view of an observer moving through space with the fluid) is

$$dv = \left(\frac{\partial v}{\partial t} \right) dt + \left(\frac{\partial v}{\partial x} \right) dx \Longrightarrow \frac{df}{dt} = 0 \tag{30.33}$$

Jeans Equations

As it stands, the collisionless Boltzmann equation is rather useless, as it is not only a function of 7 variables (many of which are not easily observable), but their derivatives. Even if $f(\vec{\omega})$ were measurable, uncertainties due to Poisson noise would play havoc with the derivatives. Fortunately, more tractable equations can be found by finding the moments (averages) of the equation. Such moments produce the Jeans Equations.

First, note that the spatial average of any quantity associated with the 6-D stellar fluid is simply found by integrating over velocity, *i.e.*,

$$\langle Q \rangle = \int Q f d^3\vec{v} \Big/ \int f d^3\vec{v} \tag{30.34}$$

Moreover, since the density of stars in the fluid, ν, is simply

$$\nu(\vec{x}) = \int f \, d^3\vec{v} \tag{30.35}$$

the spatial mean of any quantity is

$$\langle Q \rangle = \frac{1}{\nu} \int Q \cdot f d^3\vec{v} \tag{30.36}$$

So, for the first Jeans equation, start with the Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left\{ v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right\} = 0 \tag{30.31}$$

and integrate over the 3 velocity dimensions

$$\int \frac{\partial f}{\partial t} d^3v + \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} d^3v - \sum_{i=1}^{3} \int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} d^3v = 0 \tag{30.37}$$

Since time, position, and velocity are independent quantities, we can extract those dependencies from the integrals

$$\frac{\partial}{\partial t}\int f d^3v + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\int v_i f d^3v - \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i} d^3v = 0 \quad (30.38)$$

The first integral in the equation is simply the density, and the second is just the density times mean velocity, so (30.38) becomes

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3}\frac{\partial\left(\nu\langle v_i\rangle\right)}{\partial x_i} - \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i} d^3v = 0 \quad (30.39)$$

But if you expand that last integral

$$\begin{aligned}\int \frac{\partial f}{\partial v_i} d^3v &= \iiint_{-\infty}^{\infty} \frac{\partial f}{\partial v_i} dv_i dv_j dv_k \\ &= \iint dv_i dv_k \int_{-\infty}^{\infty} \partial f \\ &= \iint dv_i dv_k \, f \Big|_{-\infty}^{+\infty} = 0 \end{aligned} \quad (30.40)$$

(because obviously, the phase-space density of objects at infinite velocity is zero). This leaves us with the first Jean's equation: the 3-D equation of continuity

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3}\frac{\partial\left(\nu\langle v_i\rangle\right)}{\partial x_i} = 0 \quad (30.41)$$

For the second Jeans equation, we again start with the Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left\{ v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right\} = 0 \tag{30.31}$$

This time, however, we multiply through by v_j before integrating

$$\int v_j \frac{\partial f}{\partial t} d^3v + \sum_{i=1}^{3} \int v_i v_j \frac{\partial f}{\partial x_i} d^3v - \sum_{i=1}^{3} \int v_j \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} d^3v = 0 \tag{30.42}$$

Once again, the time and space derivatives can be taken outside the integral, leaving mean quantities

$$\frac{\partial}{\partial t} \int v_j \, f d^3v + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \int v_i v_j f d^3v - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3v = 0$$

$$\frac{\partial \left(\nu \langle v_j \rangle\right)}{\partial t} \quad + \quad \sum_{i=1}^{3} \frac{\partial \left(\nu \langle v_i v_j \rangle\right)}{\partial x_i} \quad - \quad \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3v = 0 \tag{30.43}$$

And, once again, we can expand and evaluate the individual integrals of the last term

$$\int v_j \frac{\partial f}{\partial v_i} d^3v = \iiint v_j \frac{\partial f}{\partial v_i} dv_i dv_j dv_k \tag{30.44}$$

If $j \neq i$, the result is the same as before

$$\iiint v_j \frac{\partial f}{\partial v_i} dv_i dv_j dv_k = \iint v_j dv_j dv_k \int_{-\infty}^{\infty} df = 0 \tag{30.45}$$

But if $j = i$,

$$\iiint v_i \frac{\partial f}{\partial v_i} dv_i dv_j dv_k = \iint dv_j dv_k \int_{-\infty}^{\infty} v_i df \qquad (30.46)$$

We can evaluate this last term by integrating by parts

$$\int_{-\infty}^{\infty} v_i df = v_i f \Big|_{-\infty}^{\infty} - \int f dv_i \qquad (30.47)$$

The first term is zero, and the second is just the space density, ν, so the second Jean's equation becomes

$$\frac{\partial \left(\nu \langle v_j \rangle\right)}{\partial t} + \sum_{i=1}^{3} \frac{\partial \left(\nu \langle v_i v_j \rangle\right)}{\partial x_i} + \nu \frac{\partial \Phi}{\partial x_j} = 0 \qquad (30.48)$$

The final Jeans equation comes from noting that the anisotropic pressure term (*i.e.*, the stress tensor), σ_{ij} is

$$\sigma_{i,j} = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle \qquad (30.49)$$

So, if we substitute this into the 2nd Jeans second,

$$\frac{\partial \left(\nu \langle v_j \rangle\right)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left\{ \nu \left(\sigma_{i,j}^2 + \langle v_i \rangle \langle v_j \rangle \right) \right\} + \nu \frac{\partial \Phi}{\partial x_j} = 0 \qquad (30.50)$$

If we expand this out

$$\nu \frac{\partial \langle v_j \rangle}{\partial t} + \langle v_j \rangle \frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \left\{ \frac{\partial \left(\nu \sigma_{ij}\right)}{\partial x_i} + \langle \nu_j \rangle \frac{\partial \left(\nu \langle v_i \rangle\right)}{\partial x_i} + \nu \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} \right\} = -\nu \frac{\partial \Phi}{\partial x_j} \qquad (30.51)$$

multiply the 3-D continuity equation by $\langle v_j \rangle$

$$\langle v_j \rangle \frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \langle v_j \rangle \frac{\partial \left(\nu \langle v_i \rangle \right)}{\partial x_i} = 0 \tag{30.52}$$

and subtract the two equations, we get

$$\nu \frac{\partial \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} \nu \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial \left(\nu \sigma_{ij} \right)}{\partial x_i} \tag{30.53}$$

Here, $\nu \sigma_{ij}$ is the anisotropic pressure (stress) tensor. But since σ_{ij} is symmetric, its matrix can be diagonalized. This produces the principle axes of the velocity ellipsoid.

This is the equivalent of Euler's equation for fluid flows

$$\rho \frac{\partial v}{\partial t} + \rho \left(v \cdot \nabla \right) v = \rho \nabla \Phi - \nabla p \tag{30.54}$$

where ρ is the density and p the pressure.

Jeans Equations in Cylindrical Coordinates

The Jeans and Boltzmann equation are almost never used in their Cartesian coordinate form. For most applications in spiral galaxies, cylindrical coordinates are used; in elliptical galaxies, the equations are given in spherical coordinates. The derivation of these equations is straightforward, but tedious. For the cylindrical coordinate equations, the Boltzmann equation becomes

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\theta}{R}\frac{\partial f}{\partial \theta} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\theta^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} -$$

$$\frac{1}{R}\left(v_R v_\theta + \frac{\partial \Phi}{\partial v_\theta}\right)\frac{\partial f}{\partial v_\theta} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0 \tag{30.54}$$

and if we assume azimuthal symmetry, the Jeans equations are

$$\frac{\partial \nu}{\partial t} + \frac{1}{R}\frac{\partial\, R\nu\langle v_R\rangle}{dR} + \frac{\partial\, \nu\langle v_z\rangle}{\partial z} = 0 \tag{30.55}$$

$$\frac{\partial\, \nu\langle v_R\rangle}{\partial t} + \frac{\partial\, \nu\langle v_R\rangle}{\partial R} + \frac{\partial\, \nu\langle v_R v_z\rangle}{\partial z} + \nu\left(\frac{\langle v_R^2\rangle - \langle v_\theta^2\rangle}{R} + \frac{\partial \Phi}{\partial R}\right) = 0 \tag{30.56}$$

$$\frac{\partial\, \nu\langle v_\theta\rangle}{\partial t} + \frac{\partial\, \nu\langle v_R v_\theta\rangle}{\partial R} + \frac{\partial\, \nu\langle v_\theta v_z\rangle}{\partial z} + \frac{2\nu}{R}\langle v_\theta v_R\rangle = 0 \tag{30.57}$$

$$\frac{\partial\, \nu\langle v_z\rangle}{\partial t} + \frac{\partial\, \nu\langle v_R v_z\rangle}{\partial R} + \frac{\partial\, \nu\langle v_z^2\rangle}{\partial z} + \frac{\partial\langle v_R v_z\rangle}{\partial R} + \nu\frac{\partial \Phi}{\partial z} = 0 \tag{30.58}$$

This last equation is often used in its simplified form. First, assume the galaxy is in a steady state, so that the time derivative is zero. Next, consider $\langle v_R v_z \rangle$. From symmetry, this term should be zero in the plane of the galaxy. Above and below the plane, this term *might* be zero, but worst case is that the principal axes of the stellar velocity ellipsoid are rotated to align with that of the spheroidal component. To estimate the effect this would have on $\langle v_R v_z \rangle$, we must first transform the spherical coordinate system (v_r, v_θ, v_ϕ) to the cylindrical system (v_R, v_θ, v_z). This is tedious, but straightforward. If the principle axis of the system is aligned with the disk (*i.e.*, if $\langle v_r v_\theta \rangle = 0$), the the result is

$$\begin{aligned}
\langle v_R v_z \rangle &= \left\{ \langle v_r^2 \rangle - \langle v_\theta^2 \rangle \right\} \frac{Rz}{R^2 + z^2} \\
&= \left\{ \langle v_r^2 \rangle - \langle v_\theta^2 \rangle \right\} \cdot \frac{z}{R} \frac{1}{(1 + z^2/R^2)} \\
&\approx \left\{ \langle v_r^2 \rangle - \langle v_\theta^2 \rangle \right\} \cdot \frac{z}{R} \qquad (30.59)
\end{aligned}$$

Since most of the stars are near the plane of the galaxy, $v_r \approx v_R$ and $v_\theta \approx v_z$, so

$$\langle v_R v_z \rangle \approx \left\{ \langle v_R^2 \rangle - \langle v_z^2 \rangle \right\} \cdot \frac{z}{R} \qquad (30.60)$$

Now, when we evaluate the third term of the Jeans equation

$$\begin{aligned}
\frac{\partial \langle v_R v_z \rangle}{\partial R} &\approx \frac{\partial}{\partial R} \left\{ \frac{z}{R} \left(\langle v_R^2 \rangle - \langle v_z^2 \rangle \right) \right\} \\
&\approx \frac{z}{R} \left\{ \frac{\partial \langle v_R^2 \rangle}{\partial R} - \frac{\partial \langle v_z^2 \rangle}{\partial R} - \frac{\langle v_R^2 \rangle - \langle v_z \rangle}{R} \right\} \qquad (30.61)
\end{aligned}$$

Since most of the stars in a disk galaxy are near the galactic plane, *i.e.*, $z \ll R$, this term is small. So except for the region

near the galactic center

$$\frac{\partial \langle v_R v_z \rangle}{\partial R} \approx 0 \tag{30.62}$$

Similarly, the derivative

$$\begin{aligned}
\frac{\partial \nu \langle v_R v_z \rangle}{\partial R} &\approx \frac{\partial}{\partial R} \left\{ \frac{\nu z}{R} \left(\langle v_R^2 \rangle - \langle v_z^2 \rangle \right) \right\} \\
&\approx \frac{z}{R} \left\{ \nu \frac{\partial \langle v_R^2 \rangle - \langle v_z^2 \rangle}{\partial R} + \left(\langle v_R^2 \rangle - \langle v_z^2 \rangle \right) \left(\frac{\partial \nu}{\partial R} - \frac{\nu}{R} \right) \right\} \\
&\approx 0
\end{aligned} \tag{30.63}$$

(And remember – this is the worst case scenario. The closer to the plane you are, the more likely that $\langle v_r v_z \rangle = 0$ by symmetry.) So, to first order, the Jeans equation relates the z velocity dispersion to the galactic potential by

$$\frac{\partial \Phi}{\partial z} = -\frac{1}{\nu} \frac{\partial \nu \langle v_z^2 \rangle}{\partial z} \tag{30.64}$$

To see the importance of this, we can take the derivative of this equation,

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{\partial}{\partial z} \left\{ \frac{1}{\nu} \frac{\partial}{\partial z} \left(\nu \langle v_z^2 \rangle \right) \right\} \tag{30.65}$$

and then note that near the galactic plane, the gradient of the potential is almost entirely in the z direction. Hence

$$\frac{\partial \Phi^2}{\partial z^2} \approx \nabla^2 \Phi = -\frac{\partial}{\partial z} \left\{ \frac{1}{\nu} \frac{\partial}{\partial z} \left(\nu \langle v_z^2 \rangle \right) \right\} = 4\pi G \rho \tag{30.66}$$

via Poisson's equation. Thus the mass in the disk can be measured via the z-motions of stars near the galactic plane.

Jeans Equations in Spherical Coordinates

The Boltzmann equation in spherical coordinates is

$$\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \frac{v_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial f}{\partial \phi} +$$

$$\left(\frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\right)\frac{\partial f}{\partial v_r} + \frac{1}{r}\left(v_\phi^2 \cot\theta - v_r v_\theta - \frac{\partial \Phi}{\partial \theta}\right)\frac{\partial f}{\partial v_\theta} -$$

$$\frac{1}{r}\left\{v_\phi\left(v_r + v_\theta \cot\theta\right) + \frac{1}{\sin\theta}\frac{\partial \Phi}{\partial \phi}\right\}\frac{\partial f}{\partial v_\phi} = 0 \qquad (30.67)$$

and, for steady-state systems with $\langle v_r \rangle = \langle v_\theta \rangle = 0$, the Jeans equation is

$$\frac{\partial\, \nu \langle v_r^2 \rangle}{\partial r} + \frac{\nu}{r}\left\{2\langle v_r^2 \rangle - \left(\langle v_\theta^2 + v_\phi^2\right)\right\} = -\nu \frac{\partial \Phi}{\partial r} \qquad (30.68)$$

Another way to look at this equation is to think of it in terms of hydrostatic equilibrium. Recall that the definition of pressure is

$$P = \frac{1}{3}\rho\langle v^2 \rangle = \rho \langle v_i^2 \rangle \qquad (30.69)$$

where v_i represents one component of the motion. So let P be the radial pressure imparted by the stellar motions, and Q be the tangential pressure term, *i.e.*,

$$P = \nu \langle v_r^2 \rangle \qquad Q = \nu\left(\langle v_\phi^2 \rangle + \langle v_\theta^2 \rangle\right) \qquad (30.70)$$

In this case, the Jeans equation looks is

$$\frac{dP}{dr} + \frac{2P - Q}{r} = -\nu \frac{d\Phi}{dr} \qquad (30.71)$$

If the stellar orbits are randomly distributed (*i.e.*, isotropic, so that ($\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$), then $2P = Q$, and the equation reduces to the simple hydrostatic equilibrium.

To describe the degree of orbital anisotropy in a spherical system, one often uses the parameter β,

$$\beta = 1 - \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle} \tag{30.72}$$

For isotropic orbits, $\beta = 0$; for purely radial orbits, $\beta = 1$, and for purely circular orbits, $\beta = \infty$. Note that β need not be a constant throughout the system; realistic models of spherical systems often have $\beta(r)$ as their most important free parameter.